

# Quantum Mechanics and Linearized Gravitational Waves

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The interaction of classical gravitational waves (GW) with matter is studied within a quantum mechanical framework. The classical equations of motion in the long wave-length limit is quantized and a Schroedinger equation for the interaction of GW with matter is proposed. Due to its quadrapole nature, the GW interacts with matter by producing squeezed quantum states. The resultant hamiltonian is quite different from one would expect from general principles, however. The interaction of GW with the free particle, the harmonic oscillator and the hydrogen atom is then studied using this hamiltonian.

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## §1. Introduction.

When the two words “quantum” and “gravity” appear together in the same sentence, they usually appear as “quantum gravity”, a term which immediately brings to mind energies and length scales at the level of the Planck scale. With the advent of LIGO, and the prospect of the direct detection of gravitational waves (GW) [1], [2], however, the interplay of classical and quantum gravity at a *quantum mechanical* level begins to take on added importance. Indeed, from the classical analysis the effect of GW on matter is expected to be so small that one begins to expect that a quantum mechanical treatment of this interaction is needed [3], [4]. It is, moreover, at the quantum mechanical level that experimental evidence for the interplay between gravity and quantum mechanics will first appear, rather than at the level of full quantum gravity.

Surprisingly, the interaction of GW and matter at a quantum mechanical level has not been systematically explored. We know of two approaches currently in the literature. The first is by DeWitt [5] and is concerned solely with the general formulation of quantum mechanics on a curved, background space(time). The second is by Weber [4] and, as he was concerned with the response of GW detectors to incident GW’s, deals exclusively with the interaction of GW with matter. His approach involved the direct quantization of the *linearized* classical equations of motion for a test particle interacting with a GW propagating on a flat background. Surprisingly, these two approaches are *not* equivalent to one another. One cannot, for example, obtain Weber’s result, or the results of this paper, from DeWitt’s general formalism by taking the linearized gravity limit. Fundamentally, we shall find that this is due to the different approaches taken by Weber and DeWitt. Weber begins with essentially the geodesic deviation equation which he then quantizes while DeWitt does a straightforward generalization of the quantization procedure in flat spacetime to curved spacetimes.

In this paper we shall present a systematic study of the interaction of GW with matter at a quantum mechanical level in the long wave-length and small velocity limit. Like Weber, we shall start with the classical equations of motion, but unlike his approach we shall not first linearize the equations of motion. We find that the hamiltonian derived by Weber is valid only under very restrictive circumstances and, in particular, is not valid if the test

particle is charged. In fact, due to the quadrapole nature of the GW (instead of the dipole nature of the electro-magnetic (em) field), the net effect of an incident GW on any quantum mechanical system is to produce a squeezed quantum state. This aspect of the interaction could not be seen within the linearized approach of Weber. Another aspect is that the GW is found to couple with test particle in a way that is quite similar to the minimally coupled em-field. While on the one hand this is what we would expect if the analogy between GW and em-waves is to hold, on the other hand it is quite different than what one expects from general principles. Spin-0 scalar particles like the ones we shall be concerned with in this paper are not expected to couple to the GW in this way.

The rest of this paper is organized in the following manner. In §2 we shall quantize the classical theory using the usual canonical quantization procedure. Then, in the succeeding sections §3 to §5 we shall use this hamiltonian to study the interaction of the GW with the free particle, the two dimensional harmonic oscillator, and the hydrogen atom. Concluding remarks and comparisons between DeWitt's, Weber's and our results can then be found in §6.

## §2. Quantum Mechanics.

The equations of linearized gravity on a Minkowski background are well known. From [6],

$$\Gamma_{\alpha\beta}^{\mu} = \frac{1}{2}\eta^{\mu\nu} (\partial_{\alpha}h_{\beta\nu} + \partial_{\beta}h_{\alpha\nu} - \partial_{\nu}h_{\alpha\beta}) , \quad (1)$$

where  $h_{\mu\nu}$  is the perturbation off  $\eta_{\mu\nu}$ , the flat Minkowski metric and

$$R_{\alpha\beta\mu\nu} = \frac{1}{2} (\partial_{\mu}\partial_{\beta}h_{\alpha\nu} + \partial_{\alpha}\partial_{\nu}h_{\beta\mu} - \partial_{\alpha}\partial_{\mu}h_{\beta\nu} - \partial_{\beta}\partial_{\nu}h_{\alpha\mu}) . \quad (2)$$

We shall only be concerned with GW's propagating on a flat background and its effect on scalar, spin-0 particles. By choosing the transverse and traceless gauge for the GW,

$$h_{0\mu} = 0 , \quad \nabla^{\mu}h_{\mu\nu} = 0 , \quad h_{\mu}^{\mu} = 0 , \quad (3)$$

all the gauge freedom has been removed and the GW contains only its two physical polarizations. The first polarization state  $h_{11} = -h_{22}$ ,  $h_{33} = h_{12} = h_{21} = 0$  is usually called the + polarization while the second  $h_{12} = h_{21}$ ,  $h_{jj} = 0$  (no sum) is usually called the  $\times$ -polarization. In this gauge the only non-zero components of the curvature tensor are

$$R_{0,k0}^j = -\frac{\partial \Gamma_{0k}^j}{\partial t} = -\frac{1}{2} \frac{\partial^2 h_{jk}}{\partial t^2}. \quad (4)$$

As usual, greek indices shall run from 0 – 4 while latin indices run from 1 – 3 and we are following the sign convention in [6].

Classically, the response of a scalar test particle to the passage of a gravitational wave is given by the geodesic deviation equation. The overall effect of the GW is to introduce an effective tidal force to the equations of motion,

$$m \frac{d^2 x^j}{dt^2} = -m R_{0,k0}^j x^k + F^j, \quad (5)$$

where  $x^j$  is the location of the test particle,  $m$  is its mass and  $F_j$  represents all other forces acting on the particle (see [6] for a complete derivation and explanation of the above). In the derivation of eq. (5), certain important approximations were made. First, it was assumed that the test particle is slowly moving and any velocity terms in the geodesic deviation equation may be neglected. Second, the long wave-length limit was taken, meaning that the reduced wave-length  $\lambda/2\pi$  of the GW was assumed to be very much larger than the range of motion of the test particle. The GW can then be treated as a function of time only.

Eq. (5), the classical equations of motion for the test particle, is our starting point. As we shall see, there are *two* different ways of quantizing the classical theory which are unitarily equivalent to one another only under very restrictive circumstances. We shall start with the most straightforward quantization procedure. For convenience, we shall assume that any external forces are time and velocity independent and can be represented as a potential  $V$ . Moreover, we shall always treat the GW as an external, *classical* field.

The classical lagrangian for eq. (5) is

$$L_{NR} = \frac{1}{2} m \dot{x}_j^2 - \frac{m}{2} R_{0,k0}^j x^j x^k - V, \quad (6)$$

where the dot denotes derivative with respect to time. Then choosing  $x^j$  as our generalized coordinate, the canonical momentum is  $p_j = m \dot{x}^j$ , the mechanical momentum, and the hamiltonian is

$$H_W = \frac{p_j^2}{2m} + \frac{m}{2} R_{0,k0}^j x^j x^k + V. \quad (7)$$

We may then “quantize” this hamiltonian in the usual manner by replacing  $x^j$  and  $p_j$  with operators which satisfy the canonical quantization condition:  $[x^j, p_k] = i\hbar \delta_k^j$ . It is essentially

a linearized version of eq. (7) which is used by Weber and others [7] in studying the quantum mechanical properties of GW detectors.

The second way to proceed is to note that the lagrangian eq. (6) is equivalent to

$$L = \frac{1}{2}m\dot{x}^2 - m\Gamma_{0k}^j\dot{x}_jx^k - V, \quad (8)$$

up to an integration by parts. If we once again choose  $x^j$  as our generalized coordinate, we now find that the canonical momentum is  $P_j = m\dot{x}_j - m\Gamma_{0k}^jx^k$  while the hamitonian becomes

$$H = \frac{1}{2m} \left( P_j + m\Gamma_{0k}^jx^k \right)^2 + V. \quad (9)$$

Canonical quantization then gives:  $[x^j, P_k] = i\hbar\delta_k^j$ . The Schrödinger equation is then

$$-i\hbar\frac{\partial\psi}{\partial t} = H\psi, \quad (10)$$

with the expectation value of any operator  $\mathcal{O}$  defined as

$$\langle\psi|\mathcal{O}|\psi\rangle = \int \bar{\psi}\mathcal{O}\psi d^3x. \quad (11)$$

A priori, we might have expected the presence of  $\sqrt{g^{(3)}}$  in the integral eq. (11) where  $g^{(3)}$  is the determinant of the three dimensional metric for a hypersurface. Since, however, we are dealing with *linearized* gravity,  $g_{jk}^{(3)} = \delta_{jk} + h_{jk}$  and  $g^{(3)} \approx 1 + h_{jk}^2/2$ . To lowest order,  $g^{(3)} \approx 1$ .

Since  $\Gamma_{0k}^j$  depends only on time, the quantization condition  $[x^j, P_k] = i\hbar\delta_k^j$  implies that  $[x^j, m\dot{x}_k] = i\hbar\delta_k^j$  also. The two hamiltonians eq. (7) and eq. (9) are related to one another by a simple canonical transformation. Thus there is a *time dependent* unitary transformation which maps one to the other. On this level one can just as well use either eq. (7) or eq. (9) to analyze the response of the test particle to a GW. How the system is viewed physically, on the other hand, differs tremendously from one hamiltonian to the other. Notice the striking similarity between eq. (9) and the hamiltonian for a charged particle minimally coupled to the em field. Like the em-field, the GW is expected to carry momentum, which is recognized by eq. (9), and like the vector potential for the em-field, eq. (9) implies that the connection  $\Gamma_{0k}^j$  plays a fundamental role at the quantum mechanical. One should keep in mind, however, that we are dealing with a *scalar* test particle and for a scalar particle one does not expect

the connection to appear in the hamiltonian in this manner. Instead, at the most one would expect a direct coupling to the metric of the spacetime, as was obtained by DeWitt.

Let us emphasize that this simple relationship between the two hamiltonians holds only under very restrictive conditions. For example, if the test particle is also charged, then under minimal coupling eq. (9) becomes

$$H = \frac{1}{2m} \left( P_j - \frac{q}{c} A_j + m \Gamma_{0k}^j x^k \right)^2 + V, \quad (12)$$

in the non-relativistic limit while

$$H_W = \frac{1}{2m} \left( p_j - \frac{q}{c} A_j \right)^2 + \frac{1}{2} R_{0,k0}^j x^k + V, \quad (13)$$

where  $q$  is the charge of the particle and  $A_j$  is the vector potential. Expanding out the quadratic momentum term, we find that  $H$  contains a direct interaction term between the em-field and the GW while  $H_W$  does not. Moreover, due to the presence of the em-field the canonical momentum  $P_j$  and the mechanical momentum  $m\dot{x}_j$  are no longer related to one another through a canonical transformation. The two hamiltonians will, in general, describe different physics.

Actually, this type of equivalence up to a canonical transformation between two hamiltonians also occurs when an em-field interacts with a charged particle. If an em-field in the long wave-length limit interacts with a slowly moving charged particle, then if we neglect any velocity terms in the interaction the hamiltonian can be approximated by

$$H_{em} = \frac{p_j^2}{2m} + qE_j(t)x_j(t). \quad (14)$$

where  $E_j$  is the electric field and is treated as a classical field. Because of the long wave-length approximation, is a function of time only. The charged particle actually couples minimally to the em-field, of course, but in this limit the two hamiltonians are related to one another by unitary transformation.

We should also mention that both eqs. (7) and (9) hold only in the long wave-length limit. Neither one is expected to hold in the general case where  $\Gamma_{0k}^j$  is also position dependent. In fact, if we arbitrarily allow  $R_{0,k0}^j$  to be position dependent, then the equations of motion calculated from eq. (7) will contain terms which depend on the derivative of  $R_{0,k0}^j$  which are not present in the geodesic deviation equation. Indeed, we see that since the geodesic

deviation equation depends on the curvature, the hamiltonian calculated from it can only depend on the connection and should therefore have a form which is similar to eq. (9) and not to eq. (7). We thus conclude that eq. (9) and its generalization eq. (12) are the correct hamiltonians for the interaction of a scalar test particle with a GW.

### §3. The Free Particle.

In this section we shall study the interaction of a GW with a free test particle using the hamiltonian in eq. (9). The solution of this problem is fairly standard (see, for example [8]). In general terms, due to the quadrapole nature of the GW the effect of an incident GW is to produce a squeezed quantum state, which is well known from quantum optics (see [9] for a complete review). Nevertheless, we shall present a fairly complete analysis of this system as the analysis of the interaction of the GW with the two-dimensional harmonic oscillator follows in much the same way.

We first choose the  $z$ -axis to lie parallel to  $\vec{k}$ , the wave-vector for the GW. Then, due to the transversality condition eq. (3),  $\Gamma_{0k}^j$  has non-zero componants only in the  $x - y$  plane and the particle undergoes free motion along the  $z$ -direction. The problem reduces to that of two dimensional motion with the hamiltonian

$$H = \frac{P_j^2}{2m} + \Gamma_{0k}^j x^j P_k + \frac{m}{2} \Gamma_{0k}^j \Gamma_{0l}^j x^k x^l, \quad (15)$$

where we have used the traceless condition and the indices  $j$  and  $k$  now run from 1 – 2 only. Because we are dealing with *linearized gravity*, the last term in eq. (15) dependent on  $\Gamma^2$  has little meaning and we shall set it to zero by hand.

Oftentimes we shall find it more convenient to work with  $2 \times 2$  matrices representing the two possible polarization states of the GW. When doing so, the GW is witten as a matrix  $h$  with the explicit componants of  $h$  being  $h_{jk}$  for  $j, k = 1, 2$ . Moreover, using the symmetry properties of the GW we can express

$$h_{jk} = 2f(t) \left( \varepsilon_{\times} \sigma_{jk}^1 + \varepsilon_{+} \sigma_{jk}^3 \right) = 2f(t) \varepsilon_A \sigma_{jk}^A, \quad (16)$$

where  $\sigma^1$  and  $\sigma^3$  are the first and third Pauli matrices, respectively and the index  $A$  runs from 1 – 3.  $2f(t)$  is the amplitude of the GW while  $\varepsilon_{\times}$  and  $\varepsilon_{+}$  are the polarization states of the wave. They are in general time dependent and satisfy

$$\varepsilon_{\times}^2 + \varepsilon_{+}^2 = 1, \quad (17)$$

for all  $t$ . We shall also assume that the GW incidents the particle at  $t = 0$  so that  $f(t) = 0$  for  $t \leq 0$ .

We next define raising and lowering operators

$$x^j \equiv \left( \frac{\hbar}{2m\varpi} \right)^{1/2} (a_j + a_j^\dagger) , \quad P_j \equiv -i \left( \frac{\hbar m \varpi}{2} \right)^{1/2} (a_j - a_j^\dagger) , \quad (18)$$

where  $\varpi$  is, as we shall see, related to the uncertainty in the initial position and momentum of the particle. Then,

$$H = \frac{\hbar \varpi}{2} \left( a_j^\dagger a_j + \frac{1}{2} \right) - \frac{\hbar \varpi}{4} \left( a_j^2 + a_j^{\dagger 2} \right) - \frac{i\hbar}{4} \dot{h}_{jk}(t) (a_j a_k - a_j^\dagger a_k^\dagger) . \quad (19)$$

Working in the Heisenberg representation, we find that

$$\begin{aligned} \frac{da_j}{dt} &= -i \frac{\varpi}{2} (a_j - a_j^\dagger) + \frac{1}{2} \dot{h}_{jk} a_k^\dagger , \\ \frac{da_j^\dagger}{dt} &= -i \frac{\varpi}{2} (a_j - a_j^\dagger) + \frac{1}{2} \dot{h}_{jk} a_k . \end{aligned} \quad (20)$$

Next, note that the raising and lowering operators must satisfy the commutation relations

$$[a_j(t), a_k(t)] = 0 , \quad [a_j(t), a_k^\dagger(t)] = \delta_{jk} , \quad (21)$$

at equal times. This implies that  $a_j(t)$  are related to  $a_j(0)$ , the free operators at  $t = 0$ , by the canonical transformation

$$\begin{aligned} a_j(t) &= u_{jk} a_k(0) + v_{jk} a_k^\dagger(0) , \\ a_j^\dagger(t) &= a_k^\dagger(0) \bar{u}_{kj} + a_k(0) \bar{v}_{kj} , \end{aligned} \quad (22)$$

with the bar denoting the complex conjugate. Eq. (22) is a time-dependent Bogoluibov transformation with  $u$  and  $v$  being the generalized Bogoluibov coefficients. They are  $2 \times 2$  complex matrices which, due to eq. (21), must satisfy

$$uv^t = u^t v , \quad uu^\dagger - vv^\dagger = I \quad (23)$$

written in matrix form.  $t$  denotes transpose and  $I$  is the identity matrix. Moreover, they have the boundary conditions  $u(0) = I$  and  $v(0) = 0$ . Then, from eq.(20),

$$\begin{aligned} \frac{d}{dt}(u + v^\dagger) &= -i\varpi(u - v^\dagger) + \frac{\dot{h}}{2}(u + v^\dagger) , \\ \frac{d}{dt}(u - v^\dagger) &= -\frac{\dot{h}}{2}(u - v^\dagger) . \end{aligned} \quad (24)$$



Eq. (24) is difficult to solve for general  $h_{jk}$ . For the special cases of linearly and circularly polarized GW, on the other hand, we can find exact closed form solutions to eq. (24). In the case of linearly polarized GW, the polarization states  $\varepsilon_A$  are constant while  $f(t)$  is arbitrary. One can then directly intergrate eq. (24) to obtain

$$\begin{aligned}
u_l &= \left\{ \cosh f(t) - i\frac{\varpi}{2} \left( \cosh f(t) \int_0^t \cosh(2f(t')) dt' \right. \right. \\
&\quad \left. \left. - \sinh f(t) \int_0^t \sinh(2f(t')) dt' \right) \right\} I \\
&\quad - i\frac{\varpi}{2} \left\{ \sinh f(t) \int_0^t \cosh(2f(t')) dt' \right. \\
&\quad \left. - \cosh f(t) \int_0^t \sinh(2f(t')) dt' \right\} \epsilon_A \sigma^A, \\
v_l &= i\frac{\varpi}{2} \left\{ \cosh f(t) \int_0^t \cosh(2f(t')) dt' \right. \\
&\quad \left. - \sinh f(t) \int_0^t \sinh(2f(t')) dt' \right\} I + \\
&\quad \left\{ \sinh f(t) + i\frac{\varpi}{2} \left( \sinh f(t) \int_0^t \cosh(2f(t')) dt' \right. \right. \\
&\quad \left. \left. - \cosh f(t) \int_0^t \sinh(2f(t')) dt' \right) \right\} \epsilon_A \sigma^A.
\end{aligned} \tag{25}$$

For circularly polarized GW, on the other hand,  $f(t) = f_0$ , a constant, for  $t > 0$  while the polarization states now varies with time

$$\frac{d\epsilon_+}{dt} = \Omega \epsilon_{\times}, \quad \frac{d\epsilon_{\times}}{dt} = -\Omega \epsilon_+. \tag{26}$$

The exact solution to eq. (24) in this case is quite complicated and we shall present here only the small  $f_0$  solution

$$\begin{aligned}
u_{cir} &= \left\{ 1 - \frac{i\varpi t}{2} + \frac{f_0^2}{2} (\cosh \Omega t - \cos \Omega t) \right\} I \\
&\quad - \frac{f_0}{2} \left\{ \cosh \Omega t + \cos \Omega t - 2 - i\varpi t + i\frac{\varpi}{\Omega} \sin \Omega t \right\} \vec{\varepsilon} \cdot \vec{\sigma} \\
&\quad + \frac{f_0}{2} \{ \sinh \Omega t - \sin \Omega t \} \frac{\dot{\vec{\varepsilon}} \cdot \vec{\sigma}}{\Omega} \\
&\quad + \frac{f_0^2}{2} \{ \sinh \Omega t + \sin \Omega t - 2\Omega t \} i \frac{\vec{\varepsilon} \times \dot{\vec{\varepsilon}}}{\Omega} \cdot \vec{\sigma}, \\
v_{cir} &= \left\{ \frac{i\varpi t}{2} + \frac{f_0^2}{2} (\cosh \Omega t + \cos \Omega t - 2) \right\} I
\end{aligned}$$

$$\begin{aligned}
& -\frac{f_0}{2} \left\{ \cosh \Omega t - \cos \Omega t - 2 + i\varpi t - i\frac{\varpi}{\Omega} \sin \Omega t \right\} \vec{\varepsilon} \cdot \vec{\sigma} \\
& + \frac{f_0}{2} \{ \sinh \Omega t + \sin \Omega t \} \frac{\dot{\vec{\varepsilon}} \cdot \vec{\sigma}}{\Omega} \\
& - \frac{f_0^2}{2} \{ \sinh \Omega t - \sin \Omega t \} i \frac{\vec{\varepsilon} \times \dot{\vec{\varepsilon}}}{\Omega} \cdot \vec{\sigma},
\end{aligned} \tag{27}$$

where we have used the vector notation defined in **Appendix A**. We refer the reader to **Appendix A** for the derivation of the exact solution.

To complete the quantum mechanical analysis, we note that the unitary evolution of  $a_j(t)$  can be implimented by the following unitary transformation on  $a_j(0)$  and  $a_j^\dagger(0)$ ,

$$a_j(t) = R(t)S(t)a_j(0)S^\dagger(t)R^\dagger(t), \quad a_j^\dagger(t) = R(t)S(t)a_j^\dagger(0)S^\dagger(t)R^\dagger(t), \tag{28}$$

where

$$S(t) = \exp\{(\bar{\rho}_{jk}a_j(0)a_k(0) - \rho_{jk}a_j^\dagger(0)a_k^\dagger(0))/2\}, \tag{29}$$

is the generalization of the squeeze operator of quantum optics, while

$$R(t) = \exp\{-i\epsilon_{jk}a_j^\dagger(0)a_k(0)\}, \tag{30}$$

is the generalized rotation operator.  $\rho$  and  $\tilde{\epsilon}$  are  $2 \times 2$  matrices with  $\rho$  symmetric:  $\rho = \rho^t$ . It is then straightforward to varify that  $S$  is indeed unitary while  $R$  is unitary if and only if  $\tilde{\epsilon} = \tilde{\epsilon}^\dagger$  is hermitian. The time evolution operator is then  $U(t) = R(t)S(t)$ .

Using the canonical commutation relations, we find that  $u$  and  $v$  must be related to  $\rho$  and  $\tilde{\epsilon}$  through

$$\begin{aligned}
u &= \left( \sum_{n=0}^{\infty} \frac{(\rho\rho^\dagger)^n}{(2n)!} \right) e^{i\tilde{\epsilon}}, \\
v &= \left( \sum_{n=0}^{\infty} \frac{(\rho\rho^\dagger)^n}{(2n+1)!} \right) \rho e^{-i\tilde{\epsilon}}.
\end{aligned} \tag{31}$$

The constraints eq. (23) requires that  $\tilde{\epsilon}$  also be a real matrix, while  $[\rho, \rho^\dagger] = 0$ .  $\rho$  and  $\tilde{\epsilon}$  are the generalization of the squeeze parameter and the rotation angle of quantum optics [9] with  $\rho$  containing within it the squeeze angle also.

The system has now essentially be solved and one need only specify the initial state of the particle. This involves specifying not only the initial average position  $\langle x^j(0) \rangle$  and

momentum  $\langle p^j(0) \rangle$  of the particle, but also its initial uncertainty in either position and momentum. Since  $\Delta x(0) = \sqrt{\hbar/(2m\varpi)}$  and  $\Delta p(0) = \sqrt{\hbar m\varpi/2}$ , doing so determines  $\varpi$ .

#### §4. The Harmonic Oscillator.

We now consider a GW incident on a two dimensional harmonic oscillator. Once again for simplicity we shall assume that GW travels in a direction perpendicular to the oscillating plane of the oscillator, so that

$$H = \frac{P_j^2}{2m} + \frac{1}{2}m\omega^2 x_j^2 + \Gamma_{0k}^j x^k p_j, \quad (32)$$

where  $\omega$  is the oscillation frequency of the pendulum,  $m$  is its mass and  $j, k = 1, 2$ . The oscillator is assumed to have no equilibrium length in distinct contrast to the system considered by Weber. We have once again neglected the  $\Gamma^2$  term.

We again define raising and lowering operators as in eq. (18), but with  $\omega$  instead of  $\varpi$ , and obtain

$$H = \hbar\omega(a_j^\dagger a_j + 1) - \frac{i\hbar}{2}h_{jk}(a_j a_k - a_j^\dagger a_k^\dagger). \quad (33)$$

Note that when  $\varepsilon_+ = 1$  and  $\varepsilon_- = 0$ , the  $x$  and  $y$  directions decouple and we have the hamiltonian for a one mode squeeze state (see [9]) in each direction separately. When, on the other hand,  $\varepsilon_+ = 0$  and  $\varepsilon_- = 1$ , the two directions couple and we have the hamiltonian for a two mode squeeze state.

Proceeding as before, we look for solutions of the Heisenberg equations of motion with the form eq. (22). We again obtain a set of matrix differential equations for  $u$  and  $v$ , but unlike the case of the free test particle, these equations are no-longer easily solveable even in the case of linearly and circularly polarized GW.

Let us first consider the case of linearly polarized GW for which  $h_{jk} = 2f(t)\epsilon_A\sigma_{jk}^A \equiv 2f(t)\tilde{\sigma}_{jk}$  and  $\epsilon_A$  a constant. The equations we now have to solve are

$$\frac{du}{dt} = -i\omega u + f\tilde{\sigma}v^\dagger, \quad \frac{dv}{dt} = -i\omega v + f\tilde{\sigma}u^\dagger, \quad (34)$$

along with their corresponding complex conjugate equations. Although under certain instances these equations can be solved exactly in terms of Mathieu functions, doing so will not be physically illuminating. Rather, we note that usually  $|f(t)| \ll 1$  and we can solve eq. (34) perturbatively about its  $f(t) = 0$  solution

$$\begin{aligned}
u_l &\approx e^{-i\omega t} \left( 1 + \frac{f^2(t)}{2} + 4\omega^2 \int_0^t f(t') \int_0^{t'} f(t'') e^{2i\omega(t'-t'')} dt' dt'' \right. \\
&\quad \left. + 2i\omega f(t) \int_0^t f(t') e^{2i\omega(t-t')} dt - 2i\omega \int_0^t f^2(t') dt' \right) I, \\
v_l &\approx e^{i\omega t} \left( f(t) - 2i\omega \int_0^t f(t') e^{-2i\omega(t-t')} dt' \right) \tilde{\sigma}.
\end{aligned} \tag{35}$$

As we have mentioned, the hamiltonian for the interaction of a GW with a harmonic oscillator is very similar to that of the hamiltonian for squeezed states in quantum optics. Indeed, if we solve eq. (31) in this case, we find that  $\rho = r e^{2i\phi} \tilde{\sigma}$  and  $\tilde{\epsilon} = \epsilon I$  where

$$\begin{aligned}
\tan(\epsilon - \omega t) &= 2\omega f(t) \int_0^t f(t') \cos(2\omega(t-t')) dt' - 2\omega \int_0^t f^2(t') dt' \\
&\quad + 4\omega \int_0^t f(t') \int_0^{t'} f(t'') \sin(2\omega(t'-t'')) dt' dt'', \\
\tan(\epsilon - 2\phi + \omega t) &= \frac{2\omega \int_0^t f(t') \sin(2\omega(t-t')) dt'}{f(t) - 2\omega \int_0^t f(t') \sin(2\omega(t-t')) dt'}.
\end{aligned} \tag{36}$$

There is, however, some ambiguity in the solution for  $r$ . If one uses the equation for  $u$ , one finds that

$$\begin{aligned}
r^2 &= f^2(t) - 4\omega f(t) \int_0^t f(t') \sin(2\omega(t-t')) dt' \\
&\quad + 8\omega^2 \int_0^t f(t') \int_0^{t'} f(t'') \cos(2\omega(t'-t'')) dt' dt'',
\end{aligned} \tag{37}$$

while if we use the equation for  $v$ ,

$$r^2 = f^2(t) - 4\omega f(t) \int_0^t f(t') \sin(2\omega(t-t')) dt' + 8\omega^2 \left( \int_0^t f(t') \cos(2\omega(t-t')) dt' \right)^2. \tag{38}$$

This ambiguity arises from our perturbation scheme and the desire to polar decompose  $u$  and  $v$  into a modulus and phase. The differences between the two expressions only occur in the last term, however, and are very small for  $\omega t \ll 1$ . Any ambiguity disappears if we use eq. (31) in eq. (34) to obtain differential equations for  $\rho$  and  $\tilde{\epsilon}$  and then perform the perturbative analysis on  $\rho$  and  $\tilde{\epsilon}$  directly. For our purposes, however, the perturbative solutions eqs. (36) and (37) are sufficient.

The parameters  $r$ ,  $\epsilon$  and  $\phi$  are usually called the squeeze parameter, the rotation angle and the squeeze angle, respectively. There are two properties of  $r$  and  $\phi$  that are of interest. First, from eq. (35) we expect a resonance to occur when the frequency of the GW is equal to  $2\omega$ . This can be seen explicitly by taking  $f(t) = f_0 \sin \Omega t$  for a monochromatic GW

and performing the integral in eq. (35) explicitly. Physically, this is due to the quadrupole nature of the GW, and its quadratic coupling to the harmonic oscillator. At this order of the perturbative expansion we get frequency doubling, and thus a resonance at  $\Omega = 2\omega$  for  $u$  and  $v$ . If we were to then go to the next perturbative order, we would get frequency quadrupling and a resonance at  $\Omega = 4\omega$  and so on.

Second, notice that to this order  $\phi$  is independent of the amplitude of the GW. Consequently, a very weak GW can still produce a large response in the squeeze angle. Indeed, for the monochromatic wave  $f(t) = f_0 \sin \Omega t$ ,

$$\tan(\epsilon - 2\phi + \omega t) = -2\omega \frac{\omega \sin \Omega t - \Omega \sin 2\omega t}{(\Omega^2 - 2\omega^2) \sin \Omega t - 2\omega\Omega \sin 2\omega t}, \quad (39)$$

when  $\Omega \neq 2\omega$ . Note that the denominator of eq. (39) may vanish for certain  $t$ . (This will, in fact, almost certainly happen when if the incident GW is on resonance  $\Omega = 2\omega$ .) Consequently, we can expect very large variations in the squeeze angle  $\phi$  no matter how weak the incident GW is. Since, however, the squeeze parameter  $r \sim f(t)$ , this large phase shift will still be difficult to see. Note also that this does *not* happen for  $\epsilon$ , which depends on  $f^2$  and to lowest order is unaffected by the GW.

Finally, let us consider a circularly polarized GW for which

$$\frac{du}{dt} = -i\omega u + f_0 \dot{\epsilon}_A \sigma^A v^\dagger, \quad \frac{dv}{dt} = -i\omega v + f_0 \dot{\epsilon}_A \sigma^A u^\dagger. \quad (40)$$

Although these equations can be solved exactly using the techniques described in **Appendix A**, the final result will not be useful as it involves the solution of a fourth order polynomial equation. We shall instead once again solve eq. (40) perturbatively and obtain

$$\begin{aligned} u_{cir} = e^{-i\omega t} & \left\{ 1 - 2i\omega f_0 \int_0^t \vec{\epsilon}(t) \cdot \vec{\epsilon}(t') e^{2i\omega(t-t')} dt' \right. \\ & + 4\omega^2 \int_0^t \int_0^{t'} \vec{\epsilon}(t') \vec{\epsilon}(t'') e^{2i\omega(t'-t'')} dt' dt'' \Big\} I \\ & + e^{-i\omega t} f_0^2 \left\{ 2i \frac{\omega}{\Omega} \int_0^t \frac{\vec{\epsilon}(t) \times \dot{\vec{\epsilon}}(t)}{\Omega} \cdot \vec{\epsilon}(t') e^{2i\omega(t-t')} dt' \right\} \frac{\dot{\vec{\epsilon}} \cdot \vec{\sigma}}{\Omega} \\ & + e^{-i\omega t} f_0^2 \left\{ -\Omega t + -2i \frac{\omega}{\Omega} \int_0^t \dot{\vec{\epsilon}}(t) \cdot \vec{\epsilon}(t') e^{2i\omega(t-t')} dt' \right. \\ & + 4i \frac{\omega^2}{\Omega^2} \int_0^t \int_0^{t'} \left( \vec{\epsilon}(t) \cdot \vec{\epsilon}(t') \dot{\vec{\epsilon}}(t) \cdot \vec{\epsilon}(t'') \right. \end{aligned}$$

$$\begin{aligned}
& -\vec{\varepsilon}(t) \cdot \vec{\varepsilon}(t'') \vec{\varepsilon}(t) \cdot \vec{\varepsilon}(t') \Big) e^{2i\omega(t'-t'')} dt' dt'' \Big\} i \frac{\vec{\varepsilon} \times \dot{\vec{\varepsilon}}}{\Omega} \cdot \vec{\sigma} \\
v_{cir} = & f_0 e^{i\omega t} \left( 1 + 2i\omega \int_0^t \vec{\varepsilon}(t) \vec{\varepsilon}(t') e^{-2i\omega(t-t')} dt' \right) \vec{\varepsilon} \cdot \vec{\sigma} \\
& + 2if_0 \frac{\omega}{\Omega} e^{i\omega t} \left( \int_0^t \vec{\varepsilon}(t) \cdot \vec{\varepsilon}(t') e^{-2i\omega(t-t')} dt' \right) \frac{\dot{\vec{\varepsilon}} \cdot \vec{\sigma}}{\Omega} .
\end{aligned} \tag{41}$$

We once again expect a resonance at  $\Omega = 2\omega$ . The corresponding  $\rho$  and  $\tilde{\varepsilon}$  can also be found in this case.

## §5. The Hydrogen Atom.

As we shall be using time dependent perturbation theory to analyze the interaction of GW with the hydrogen atom, in this section we shall keep to generalities. An exposition of time dependent perturbation theory can be found in any quantum mechanics textbook (see for example [10]). From eq. (12) the relevant hamiltonian is

$$\begin{aligned}
H = & \frac{p_j^2}{2\mu} + V - \frac{e}{\mu c} A_j p_j + \frac{e^2}{2\mu c^2} A_j^2 \\
& - \frac{e}{c} \Gamma_{0k}^j A_j x^k + \Gamma_{0k}^j x^k p_j + \frac{m}{2} \Gamma_{0k}^j \Gamma_{0l}^j x^k x^l
\end{aligned} \tag{42}$$

where  $\mu$  is the reduced mass of the hydrogen atom,  $A_j$  is the vector potential and  $V$  is the usual Coulomb potential for the hydrogen atom. Although the vector potential will be treated as a field operator, the energy density for the em-field has not been included in eq. (42). We have moreover neglected the spin of the electron, as we have not yet considered the effect of the GW on the spin of a particle. We also note that even at atomic energies, the wave-length of the GW will still be much larger than the size of the hydrogen atom and the long wave-length approximation is still valid.

The first four terms in eq. (42) are what one usually obtains for a minimally coupled hydrogen atom. The additional three terms comes from the interaction of the GW with the atom and we shall treat these terms perturbatively. As usual, since the last term involves  $\Gamma^2$  we shall not consider its affects.

Consider first the interaction term

$$H_{int}^{(1)} = \frac{1}{2} \hbar_{jk} x^k p_j , \tag{43}$$

which couples the GW directly to the electron and can cause either the absorption or emission of a “graviton”. From time-dependent perturbation theory the transition probability

between an initial state  $|init\rangle$  and a final state  $|fin\rangle$  is

$$|c(+\infty)|^2 = \frac{\mu^2 \pi^2}{4\hbar^2} \omega_{if}^4 |\hat{h}_{jk}(\omega_{if})|^2 |\langle fin | (x_j x_k - \delta_{jk} x^2/3) | init \rangle|^2 \quad (44)$$

where  $\hat{h}_{jk}(\omega_{if})$  is the fourier transform of  $h_{jk}(t)$  and  $\hbar\omega_{if}$  is the energy difference between the initial and final states. The quadrapole nature of the interaction can now be seen explicitly. From the Wigner-Eckhart theorem the allowed transitions are:  $\Delta m = \pm 2$ ,  $\Delta l = \pm 2$  where  $m$  and  $l$  are the angular momentum quantum numbers.

Using eq. (44) and the fact that the intensity of the GW is  $I = h_{jk}^2 \omega^2 c^3 / 32\pi G$ , find that the cross-section for this interaction has the form

$$\sigma_{cross}(\omega) = J_{if} l_p^2, \quad (45)$$

where  $J_{if}$  is a numerical factor depending on the initial and final states and  $l_p^2 = G\hbar/c^3$  is the Planck length. (The reader should not confuse this  $\sigma_{cross}$  with the Pauli matrices.) The cross-section is thus proportional to  $l_p^2 = 10^{-66} \text{ cm}^2$ , and is extremely small no matter what value  $J_{if}$  takes. Combined with the expectation that the intensity of GW's in the universe which are at atomic energies are very low, such a small cross-section means that it will essentially be impossible to see a GW induced transition in a hydrogen atom. Nevertheless, it is interesting to see the appearance of the Planck scale even at this simplistic level. Notice also the absence of knowledge of the em nature of the system, such as the charge of either the electron or the nucleus, in eq. (45). This once again underscores the fact that  $H_{int}^{(1)}$  is due to the direct interaction of gravity with matter.

Let us next consider the interaction term

$$H_{int}^{(2)} = \frac{e}{2c} \dot{h}_{jk} A_j x^k. \quad (46)$$

In many ways this is much more interesting as it couples the GW with both the em-field and matter. In particular, this is a *dipole* coupling in contrast to the quadrapole coupling of  $H_{int}^{(1)}$ . The absorption of a graviton can thus cause the emission of a photon along with the excitation of the hydrogen atom to an excited state. Conversely, the emission of a photon can also cause the emission of a graviton. Theoretically, the passage of a GW should shorten the lifetime of the any excited state of the atom. Indeed, if we combined eq. (46) with the usual dipole interaction term in eq. (42), we find that the cross-section for emission or absorption

of a photon in the dipole approximation will be slightly larger by a factor  $1 + \text{const.} h_{jk}^2(0)$  where the constant depends on the polarization state of the GW. Since, however,  $h_{jk}$  is extremely small, this shortening cannot be seen experimentally.

Using Fermi's Golden Rule, we can also calculate the transition cross-section for the emission or absorption of a single photon due to  $H_{int}^{(2)}$ ,

$$\sigma_{cross}(\omega) = J'_{if} \alpha (\omega_{if} a_{bohr}/c)^2 l_p^2, \quad (47)$$

where once again  $J'_{if}$  is a numerical factor depending on the initial and final states of the hydrogen atom,  $\hbar\omega_{if}$  is the energy of the photon,  $a_{bohr}$  is the Bohr radius and  $\alpha$  is the fine structure constant. For typical atomic transitions,  $\sigma_{cross} \sim 10^{-5} l_p^2$  which is even smaller than that of the direct interaction  $H_{int}^{(1)}$  since the transition is mediated by the emission of a photon.

## §7. Concluding Remarks.

To conclude, we have quantized the classical equations of motion for a scalar, spin-0 interacting with a classical GW in the long wavelength limit when the velocities of the particle is small. We find that due to the quadrapole nature of the interaction the effect of the GW is to produce a squeezed quantum state <sup>1</sup>. We have then used this formalism to calculate the effects of the GW on the free particle, the harmonic oscillator and the hydrogen atom.

The hamiltonian  $H$  that we have obtained is quite peculiar, however. On the one hand it has the form one would expect if the analogy between the GW and the em-wave is to hold at this level. Like the em-wave, the GW carries momentum and energy. And like the em-wave, since the classical equations of motion depends on a physical observable (the field strength for the em-wave, the curvature tensor for the GW), the hamiltonian must depend on their respective connections. A coupling of the form  $(P_j + m\Gamma_{0k}^j x^k)^2$  would thus seem quite natural. Moreover, like the vector potential for the em-wave it means that the connection  $\Gamma_{0k}^j$  now takes on an independent meaning at the quantum mechanical level. On

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<sup>1</sup>Curiously, the creation of primordial GW from the de Sitter vacuum is also due to quantum mechanical squeezing [11].



the other hand, this coupling resembles minimal coupling for the gravitational field. As we are dealing with a scalar particle, from general principles alone we would not have expected a coupling of this form. We would instead have expected a hamiltonian of the form given by DeWitt [5]. One cannot, however, obtain  $H$  directly from DeWitt's hamiltonian by taking the linearized gravity limit.

DeWitt in [5] considered the quantum mechanics of a particle constrained to move on a curved space(time). Then if  $g_{ij}$  is the metric *on the three dimensional hypersurface* he obtained a hamiltonian of the form

$$H_{DW} = \frac{1}{2m} g^{jk} \left( -i\hbar \frac{\partial}{\partial x^j} - \frac{i\hbar}{4} \frac{\partial \ln g}{\partial x^j} \right) \left( -i\hbar \frac{\partial}{\partial x^k} - \frac{i\hbar}{4} \frac{\partial \ln g}{\partial x^k} \right) \quad (48)$$

for a non-relativistic particle where  $g$  is the determinant of  $g_{jk}$ .<sup>2</sup> Let us now try to apply this hamiltonian to the case of GW. In this case  $g_{jk} = \delta_{jk} + h_{jk}$  and due to the traceless condition on  $h_{jk}$ ,  $g = 1$  up to first order in  $h_{jk}$ . Consequently,

$$H_{DW} \approx -\frac{\hbar^2}{2} \left( \frac{\partial}{\partial x^j} \right)^2 - \frac{\hbar^2}{2} h^{jk} \frac{\partial}{\partial x^j} \frac{\partial}{\partial x^k}. \quad (49)$$

This hamiltonian is very different from only the one we have derived, but also from the one used by Weber. Moreover, from the transformation law for general coordinate transformations (eq. (5.42) of [5]),  $H_{DW}$  cannot be mapped to  $H$  through a judicious coordinate transformation.

We should also mention that in DeWitt's derivation of  $H_{DW}$  he actually obtained  $p_j = -i\hbar(\partial_j + \partial_j \ln g/4) - \partial_j \Phi$  where  $\Phi$  is an arbitrary real function of  $x_j$  and  $t$  (eq. (5.24) of [5]).  $\Phi$  was then set to zero by arguing that it can always be removed by a *local* unitary phase transformation on the wavefunction. One finds precisely a term of this form in  $H$ . This, however, still does not explain the presence of the second term in  $H_{DW}$  which is absent in  $H$ .

We believe that the differences between  $H_{DW}$  and  $H$  to be more fundamental than a simple gauge transformation of the wavefunction, however. DeWitt's derivation of  $H_{DW}$  was based on a natural generalization of quantum mechanics on flat spacetime to curved

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<sup>2</sup>DeWitt considered motion on an  $n$ -dimensional space. We need not be so general.

spacetimes. Our hamiltonian, on the other hand, is based on the classical geodesic *deviation* equation. Physically, this is because the passage of the GW will affect not only the test particle, but also the observer (measuring apparatus, laboratory setup). Since the observer is also part of the universe, it cannot be isolated from any gravitational effects and also responds to the GW. Consequently, one cannot measure (observe) the *absolute* motion of the test particle, but rather the *relative* motion between the observer and the test particle. This motion is governed by the geodesic deviation equation. In DeWitt’s formalism it is not certain where the observer is. Since he deals with only a single particle, we suspect that the observer has tacitly been “removed” from the spacetime, something one can do in Newtonian mechanics, but cannot do in general relativity. As the results of this paper is only valid within the long wave-length limit and for a fixed choice of gauge we have not yet been able to show this explicitly.

Further study of the interaction of GW with matter is clearly needed to clarify these and other issues. In particular, a version of the hamiltonian which is valid outside of the long wave-length approximation is needed and the effect of GW on particles with spin should also be incorporated [12]. Once this is accomplished, a thorough comparison of our formalism and DeWitt’s results can be made.

We end this paper by briefly mentioning the differences between the hamiltonian used by Weber and the one we derived. As the work by Weber was mostly concerned with the properties of the interaction of GW with gravitational wave detectors, he had at their disposal a macroscopic length  $l$ . For the beam detector this is the length of one arm of the interferometer, while for the bar detector this is the equilibrium length of the spring for a harmonic oscillator. In both cases, the hamiltonian that was used has the form

$$H_{Web} = \frac{p^2}{2m} + \frac{m}{2}\omega^2 x^2 + mR_{0,10}^1 x l, \quad (50)$$

along, say, the  $x$ -direction. (The harmonic oscillator term is absent of the beam detector). Importantly,  $x$  is measured from its *equilibrium* position. If we replace  $x \rightarrow x + l$  in eq. (7) and expand out the quadratic term, we obtain  $H_{Web}$  up to a term proportional to  $x^2$ . This was dropped in comparison to  $xl$  since the response to the GW is expected to be quite small classically. While this is perfectly valid in classical dynamics, since  $x$  is now an operator doing so is somewhat questionable at a quantum mechanical level. We have, however, re-

performed the analysis of both the beam and the bar detectors using the full hamiltonian eq. (9) in a quantum mechanics framework and found very little differences in the results obtained from using  $H_{Web}$  verses  $H$ . Consequently, as long as the linearization of eq. (9) is valid, and as long as the test particle is not charged, the hamiltonian used by Weber and others gives a good description of the interaction of a GW with matter.

## APPENDIX A

We first define  $\xi = u + v^\dagger$  and  $\chi = u - v^\dagger$  so that eq. (24) becomes

$$\frac{d\xi}{dt} = -i\varpi\chi + f_0\dot{\epsilon}_A\sigma^A\xi, \quad \frac{d\chi}{dt} = -f_0\dot{\epsilon}_A\sigma^A\chi, \quad (\text{A1})$$

for circularly polarized GW. Next, we note that any  $2 \times 2$  complex matrix  $M$  can be written has  $M = \theta_0 I + \theta_A \sigma^A$  where  $\theta_0$  and  $\theta_A$  are complex numbers. We next consider  $\vec{\theta} = (\theta_1, \theta_2, \theta_3)$  as being a vector in a three dimensional complex space. Clearly the polarization states of the GW can also be represented as a vector  $\vec{\epsilon}$  in this space. Moreover,  $\vec{\epsilon}$ ,  $\dot{\vec{\epsilon}}$  and  $\vec{\epsilon} \times \dot{\vec{\epsilon}}$  are mutually orthogonal and thus form a natural coordinate system for this space. Consequently, we look for solutions of (A1) with the form

$$\chi = AI + B\vec{\epsilon} \cdot \vec{\sigma} + C\frac{\dot{\vec{\epsilon}} \cdot \vec{\sigma}}{\Omega} + Di\frac{\vec{\epsilon} \times \dot{\vec{\epsilon}}}{\Omega} \cdot \vec{\sigma}, \quad (\text{A2})$$

where  $A, B, C, D$  can be complex functions. Then

$$\begin{aligned} \frac{dA}{dt} + f_0\Omega C &= 0, \\ \frac{dB}{dt} - \Omega C - f_0\Omega D &= 0, \\ \frac{dC}{dt} + \Omega B + f_0\Omega A &= 0, \\ \frac{dD}{dt} - f_0\Omega B &= 0. \end{aligned} \quad (\text{A3})$$

Next, using the boundary condition  $\chi(0) = u(0) = I$ , we find that

$$\begin{aligned} A(x) &= \frac{\nu_+^2 \cos \nu_- x - \nu_-^2 \cos \nu_+ x + f_0^2 (\cos \nu_- x - \cos \nu_+ x)}{\nu_+^2 - \nu_-^2}, \\ B(x) &= -\frac{1}{f_0} \frac{(\nu_+^2 + f_0^2)(\nu_-^2 + f_0^2)}{\nu_+^2 - \nu_-^2} (\cos \nu_- x - \cos \nu_+ x), \end{aligned}$$

$$\begin{aligned}
C(x) &= \frac{1}{f_0} \frac{\nu_+ \nu_- (\nu_+ \sin \nu_- x - \nu_- \sin \nu_+ x) + f_0^2 (\nu_- \sin \nu_- x - \nu_+ \sin \nu_+ x)}{\nu_+^2 - \nu_-^2}, \\
D(x) &= -\frac{(\nu_+^2 + f_0^2)(\nu_-^2 + f_0^2)}{\nu_+^2 - \nu_-^2} \left( \frac{\sin \nu_- x}{\nu_-} - \frac{\sin \nu_+ x}{\nu_+} \right),
\end{aligned} \tag{A4}$$

where  $x = \Omega t$  and

$$\nu_\pm^2 \equiv \frac{1}{2} \left( 1 - 2f_0^2 \pm \sqrt{1 - 4f_0^2} \right). \tag{A5}$$

The solution for  $\xi$  follows in exactly the same way. We once again take

$$\xi = EI + F\vec{\varepsilon} \cdot \vec{\sigma} + G \frac{\dot{\vec{\varepsilon}} \cdot \vec{\sigma}}{\Omega} + Hi \frac{\vec{\varepsilon} \times \dot{\vec{\varepsilon}}}{\Omega} \cdot \vec{\sigma}, \tag{A6}$$

and now find that

$$\begin{aligned}
E(x) &= \frac{\kappa_+^2 \cosh \kappa_- x - \kappa_-^2 \cosh \kappa_+ x + f_0^2 (\cosh \kappa_+ x - \cosh \kappa_- x)}{\kappa_+^2 - \kappa_-^2} + \frac{i\varpi C}{f_0 \Omega} + \frac{i\varpi D}{f_0^2 \Omega}, \\
F(x) &= \frac{1}{f_0} \frac{(\kappa_+^2 - f_0^2)(\kappa_-^2 - f_0^2)}{\kappa_+^2 - \kappa_-^2} (\cosh \kappa_+ x - \cosh \kappa_- x) + \frac{i\varpi D}{f_0 \Omega}, \\
G(x) &= \frac{1}{f_0} \frac{\kappa_+ \kappa_- (\kappa_+ \sinh \kappa_- x - \kappa_- \sinh \kappa_+ x) + f_0^2 (\kappa_+ \sinh \kappa_+ x - \kappa_- \sinh \kappa_- x)}{\kappa_+^2 - \kappa_-^2}, \\
H(x) &= -\frac{(\kappa_+^2 - f_0^2)(\kappa_-^2 - f_0^2)}{\kappa_+^2 - \kappa_-^2} \left( \frac{\sinh \kappa_+ x}{\kappa_+} - \frac{\sinh \kappa_- x}{\kappa_-} \right),
\end{aligned} \tag{A7}$$

with

$$\kappa_\pm^2 \equiv \frac{1}{2} \left( 1 + 2f_0^2 \pm \sqrt{1 + 4f_0^2} \right). \tag{A8}$$

Notice that for  $f_0 \ll 2$ , both  $\nu_\pm$  and  $\kappa_\pm$  are real. Eq. (27) now follows if we take the  $f_0 \rightarrow 0$  limit and keep terms up to  $O(f_0^2)$ .

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